

# COMPOSITIONS OF POLYHARMONIC MAPPINGS

GANG LIU AND SAMINATHAN PONNUSAMY

**ABSTRACT.** The paper is devoted to the study of compositions of polyharmonic mappings in simply connected domains. More precisely, we determine necessary and sufficient conditions of polyharmonic mapping  $f$  such that  $f \circ F$  (resp.  $F \circ f$ ) is  $l$ -harmonic for any analytic function (or harmonic mapping but not analytic, or  $q$ -harmonic mapping but not  $(q - 1)$ -harmonic)  $F$ .

## 1. INTRODUCTION AND MAIN RESULTS

A complex-valued function  $f$  is called a harmonic mapping if it satisfies the harmonic equation  $\Delta f = 0$ , where  $\Delta$  denotes the complex Laplacian operator

$$\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Evidently, every harmonic mapping in a simply connected domain admits the representation  $f(z) = A(z) + \overline{B(z)}$ . Properties of harmonic mappings have been investigated extensively (see the monograph of Duren [15]), especially after the appearance of the pioneering article of Clunie and Sheil-Small [14] in 1984. A simple fact is that a harmonic mapping of an analytic function is harmonic, but an analytic function of a harmonic mapping is not necessarily harmonic. On the other hand, some elementary facts about compositions of certain harmonic mappings, which escape the attention of many, were addressed by Reich [24] in 1987. For instance the following result was established in [24].

**Theorem A.** *Suppose  $f(z) = z + \overline{B(z)}$ , where  $B(z)$  is analytic and  $G(z) = B'(z)$ . A necessary and sufficient condition that there locally exists a non-affine complex harmonic function  $g(w)$ , such that  $g(f(z))$  is harmonic is that  $G(z)$  satisfies*

$$(G')^2 = \alpha^2 G^4 + 2cG^3 + (\overline{\alpha})^2 G^2$$

*for some complex constant  $\alpha$  and some real constant  $c$ .*

In addition to Theorem A, several important special cases of it were also obtained in [24] by expressing the analytic functions  $G$  and  $B$  in terms of elementary functions. The present article is motivated by the work of Reich which provides the local description of all harmonic mappings  $f$  such that  $g \circ f$  is harmonic for some non-affine harmonic  $g$ . Twenty years later, properties of the composition of harmonic mappings with harmonic mappings, and the composition of biharmonic mappings with biharmonic mappings were investigated in [7]. Nevertheless, non-analytic and non-harmonic functions play significant

---

2010 *Mathematics Subject Classification.* Primary: 31A05, 31A30.

*Key words and phrases.* analytic, anti-analytic and polyharmonic.

File: LiuSamy1'2016'.tex, printed: 21-10-2016, 0.21.

role, eg. biharmonic mappings in fluid dynamics and elasticity problems (see [16, 17, 19]). However, the investigation of biharmonic mappings in the context of geometric function theory is a recent one (see [1, 2, 3, 6, 7, 13]).

We are interested in iterations of the Laplace operator, namely,  $p$ -harmonic operators defined inductively by  $\Delta^p = \Delta(\Delta^{p-1})$  for  $p = 2, \dots$ . A  $2p$ -times continuously differentiable complex-valued function  $f$  in a simply connected domain  $D \subseteq \mathbb{C}$  is called  $p$ -harmonic in  $D$  if  $f$  satisfies  $p$ -harmonic equation  $\Delta^p f = 0$  in  $D$ .  $f$  is called polyharmonic if it is  $p$ -harmonic for some  $p \in \mathbb{N}$ . For  $p = 1$  (resp. 2),  $f$  is harmonic (resp. biharmonic). Obviously, every  $p$ -harmonic mapping is  $(p + 1)$ -harmonic. It is easy to see that,  $f$  is  $p$ -harmonic in a simply connected domain  $D$  if and only if (see finite Almansi expansion [4, Proposition 1.3] and [9, Proposition 1.1])

$$(1) \quad f(z) = \sum_{k=1}^p |z|^{2(k-1)} G_k(z),$$

where each  $G_k(z)$  is harmonic in  $D$ . There is now a long list of articles in the literature on this subject. For recent results on  $p$ -harmonic mappings, we refer to the articles [8, 9, 10, 11, 12, 18, 20, 21, 22, 23]. Another motivation for the study of polyharmonic mappings is from the recent work of Borichev and Hedenmalm [5] on the study of *second order elliptic partial differential equations*  $T_\alpha(f) = 0$  in the unit disk  $\mathbb{D} = \{z : |z| < 1\}$ , where  $\alpha \in \mathbb{R}$  and

$$T_\alpha = -\frac{\alpha^2}{4}(1 - |z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1 - |z|^2)^{-\alpha-1} \left( z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right) + (1 - |z|^2)^{-\alpha} \frac{\partial^2}{\partial z \partial \bar{z}}.$$

In particular if we take  $\alpha = 2(p - 1)$ , then  $f$  satisfying  $T_\alpha(f) = 0$  is  $p$ -harmonic. Clearly, the choice  $\alpha = 0$  gives that  $f$  is harmonic. Moreover, the problem of when the composite mappings of  $p$ -harmonic mappings with a fixed analytic function are  $l$ -harmonic was discussed in [21], where  $l \in \{1, \dots, p\}$ .

In what follows, the numbers  $q, l$  are positive integers unless otherwise stated.  $[x]$  denotes the largest integer no more than  $x$ , where  $x$  is a real number. Recall that  $f(z)$  is called an affine mapping if  $f(z) = \alpha z + \bar{\beta} z + \gamma$ , where  $\alpha, \beta$  and  $\gamma$  are some constants. Similarly,  $f$  is called a harmonic polynomial of degree  $n$  if  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic polynomials such that  $n = \max\{\deg h(z), \deg g(z)\}$ .

In this paper we are mainly concerned with the properties of the composition of polyharmonic mappings. We completely solve the following problem: *What is the polyharmonic mapping if all its post-compositions by any  $q$ -harmonic mapping are  $l$ -harmonic?* Our main results follow.

**Theorem 1.** *Let  $f$  be a polyharmonic mapping. Then*

- (a) *for any analytic  $F$ ,  $f \circ F$  is  $l$ -harmonic if and only if  $f$  is harmonic;*
- (b) *for any harmonic  $F$  which is not analytic,  $f \circ F$  is  $l$ -harmonic if and only if  $f(z)$  is an affine mapping.*
- (c) *for any  $q$ -harmonic ( $q \geq 2$ )  $F$  which is not  $(q - 1)$ -harmonic,  $f \circ F$  is  $l$ -harmonic if and only if  $f$  is a harmonic polynomial of degree  $t$ , where  $t \leq \min\{1, [\frac{l-1}{q-1}]\}$ .*

**Corollary 1.** *Let  $f$  be an analytic function. Then*

- (a) for any harmonic  $F$  which is not analytic,  $f \circ F$  is  $l$ -harmonic if and only if  $f(z)$  is linear in  $z$ .
- (b) for any  $q$ -harmonic ( $q \geq 2$ )  $F$  which is not  $(q-1)$ -harmonic,  $f \circ F$  is  $l$ -harmonic if and only if  $f(z)$  is an analytic polynomial of degree  $t$ , where  $t \leq \min \{1, [\frac{l-1}{q-1}]\}$ .

We next partly solve the problem of characterizing all polyharmonic mappings if all its pre-compositions by any  $q$ -harmonic mapping are  $l$ -harmonic.

**Theorem 2.** *Let  $f$  be a harmonic mapping. Then*

- (a) for any harmonic  $F$ ,  $F \circ f$  is  $l$ -harmonic if and only if  $f$  is analytic or anti-analytic;
- (b) for any  $q$ -harmonic ( $q \geq 2$ )  $F$  which is not  $(q-1)$ -harmonic,  $F \circ f$  is  $l$ -harmonic if and only if either  $f(z)$  or  $\overline{f(z)}$  is an analytic polynomial of degree  $t$ , where  $t \leq [\frac{l-1}{q-1}]$ .

**Corollary 2.** *Let  $f$  be an analytic function. Then, for any  $q$ -harmonic ( $q \geq 2$ )  $F$  but not  $(q-1)$ -harmonic,  $F \circ f$  is  $l$ -harmonic if and only if  $f(z)$  is an analytic polynomial of degree  $t$ , where  $t \leq [\frac{l-1}{q-1}]$ .*

**Theorem 3.** *Let  $f$  be a polyharmonic mapping. Then*

- (a) for any harmonic  $F$ ,  $F \circ f$  is harmonic (or biharmonic) if and only if  $f$  is analytic or anti-analytic;
- (b) for any  $q$ -harmonic ( $q \geq 2$ )  $F$  which is not  $(q-1)$ -harmonic,  $F \circ f$  is harmonic if and only if  $f(z)$  is identically constant;
- (c) for any  $q$ -harmonic ( $q \geq 2$ )  $F$  which is not  $(q-1)$ -harmonic,  $F \circ f$  is biharmonic if and only if either  $f(z)$  or  $\overline{f(z)}$  is an analytic polynomial of degree  $t$ , where  $t \leq [\frac{1}{q-1}]$ .

The cases where there exist difficulties are formulated as an open problem.

**Conjecture 1.** *Let  $f$  be a polyharmonic mapping. Then*

- (a) for any harmonic  $F$ ,  $F \circ f$  is  $l$ -harmonic if and only if  $f$  is analytic or anti-analytic;
- (b) for any  $q$ -harmonic ( $q \geq 2$ )  $F$  which is not  $(q-1)$ -harmonic,  $F \circ f$  is  $l$ -harmonic if and only if either  $f(z)$  or  $\overline{f(z)}$  is an analytic polynomial of degree  $t$ , where  $t \leq [\frac{l-1}{q-1}]$ .

## 2. SOME NOTATIONS AND PRELIMINARY RESULTS

For simplicity, we introduce the following notations. For  $p \in \mathbb{N}$ , we let

$$\mathbb{H}_p = \{f : f \text{ is } p\text{-harmonic}\},$$

and  $\mathbb{H}_0 = \{f : f \text{ is analytic}\}$ . Set  $\mathbb{H}_p^* = \mathbb{H}_p \setminus \mathbb{H}_{p-1}$  for  $p \geq 1$  and  $\mathbb{H}_0^* = \mathbb{H}_0$ . Obviously,

$$\mathbb{H}_p = \cup_{i=0}^p \mathbb{H}_i^* \quad \text{and} \quad \mathbb{H}_i^* \cap \mathbb{H}_j^* = \emptyset \quad (i \neq j).$$

We observe that if  $f$  is  $p$ -harmonic and is represented by (1) with  $G_p(z) \neq 0$  for  $p \geq 2$ , then  $f \in \mathbb{H}_p^*$ . Finally, we introduce

$$\mathbb{H}_{p,q}^l = \{f : f \in \mathbb{H}_p \text{ such that } F \circ f \in \mathbb{H}_l \text{ for any } F \in \mathbb{H}_q^*\},$$

where  $p, q, l$  are non-negative integers. Clearly, constant functions belong to  $\mathbb{H}_{p,q}^l$ .

**Proposition 1.** *We have the following properties.*

- (a)  $\mathbb{H}_{p,q}^l \subseteq \mathbb{H}_{p+1,q}^l$  and  $\mathbb{H}_{p,q}^l \subseteq \mathbb{H}_{p,q}^{l+1}$ ;
- (b)  $\mathbb{H}_{p,q}^l \subseteq \mathbb{H}_{p,q-1}^l \subseteq \cdots \subseteq \mathbb{H}_{p,1}^l = \mathbb{H}_{p,0}^l$ ;
- (c)  $\mathbb{H}_{p,q}^l = \mathbb{H}_{l,q}^l$  ( $l \leq p$ ).

*Proof.* (a) The proofs of the inclusions in (a) are trivial.

(b) For the chain of inclusions in (b), we first show that  $\mathbb{H}_{p,q+1}^l \subseteq \mathbb{H}_{p,q}^l$  for any  $q \in \mathbb{N}$ . Assume that  $f \in \mathbb{H}_{p,q+1}^l$ . Then  $f \in \mathbb{H}_p$  and  $F_{q+1} \circ f \in \mathbb{H}_l$  for any  $F_{q+1} \in \mathbb{H}_{q+1}^*$ . Also, for any  $F_q \in \mathbb{H}_q^*$ , it is easy to see that  $F_{q+1} + F_q \in \mathbb{H}_{q+1}^*$  and

$$\Delta^l(F_q \circ f) = \Delta^l((F_{q+1} + F_q) \circ f) - \Delta^l(F_{q+1} \circ f) = 0.$$

Thus,  $f \in \mathbb{H}_{p,q}^l$  and hence,  $\mathbb{H}_{p,q+1}^l \subseteq \mathbb{H}_{p,q}^l$ .

For the completion of the proof of the inclusions in (b), we only need to show that  $\mathbb{H}_{p,1}^l \supseteq \mathbb{H}_{p,0}^l$ . To do this, we let  $F \in \mathbb{H}_1^* = \mathbb{H}_1 \setminus \mathbb{H}_0$ . Then,  $F$  is harmonic with the representation  $F = h + \bar{g}$ , where  $h$  and  $g$  are analytic, but  $g$  is not a constant function. Next, we assume  $f \in \mathbb{H}_{p,0}^l$ . Then  $h \circ f \in \mathbb{H}_l$  and  $g \circ f \in \mathbb{H}_l$ . Therefore,  $F \circ f = h \circ f + \overline{g \circ f} \in \mathbb{H}_l$  and thus,  $f \in \mathbb{H}_{p,1}^l$  which shows that  $\mathbb{H}_{p,1}^l \supseteq \mathbb{H}_{p,0}^l$ .

(c) We claim that  $\mathbb{H}_{p,0}^l = \mathbb{H}_{l,0}^l$  ( $l \leq p$ ). First, we prove that  $\mathbb{H}_{p,0}^l \subseteq \mathbb{H}_{l,0}^l$ , since  $\mathbb{H}_{p,0}^l \supseteq \mathbb{H}_{l,0}^l$  by (a). Next, we assume that  $f \in \mathbb{H}_{p,0}^l$ . Then  $f \in \mathbb{H}_p$  and  $F \circ f \in \mathbb{H}_l$  for any  $F \in \mathbb{H}_0$ . Choosing  $F(z) = z$ , we see that  $f = F \circ f \in \mathbb{H}_l \subseteq \mathbb{H}_p$ . Thus,  $f \in \mathbb{H}_{l,0}^l$  and thus,  $\mathbb{H}_{p,0}^l \subseteq \mathbb{H}_{l,0}^l$ .

Next we show that  $\mathbb{H}_{p,q}^l = \mathbb{H}_{l,q}^l$  ( $l \leq p$ ). Since  $\mathbb{H}_{p,q}^l \subseteq \mathbb{H}_{p,0}^l = \mathbb{H}_{l,0}^l$  by (b), it follows that  $\mathbb{H}_{p,q}^l \subseteq \mathbb{H}_{l,q}^l$ . Thus,  $\mathbb{H}_{p,q}^l = \mathbb{H}_{l,q}^l$  because of the inclusion  $\mathbb{H}_{l,q}^l \subseteq \mathbb{H}_{p,q}^l$  by (a).  $\square$

**Proposition 2.** *We have  $\mathbb{H}_{2,0}^2 = \{f : f \text{ is either analytic or anti-analytic}\}$ .*

*Proof.* Set  $F_m(z) = e^{mz}$  ( $m \neq 0$ ). Assume that  $f \in \mathbb{H}_{2,0}^2$ . Then  $f \in \mathbb{H}_2$  and  $F_m \circ f(z) \in \mathbb{H}_2$ , which shows that  $\Delta^2(e^{mf}) = 0$ . By computation, we have that

$$\Delta(e^{mf}) = 4me^{mf}(f_z \bar{z} + m f_z f_{\bar{z}}) \quad \text{and} \quad \Delta^2(e^{mf}) = 16m^2 e^{mf} A_m(z),$$

where

$$A_m(z) = 2(f_z^2 \bar{z} + f_z f_{\bar{z}^2} + f_{\bar{z}} f_{z^2} \bar{z}) + f_{z^2} f_{\bar{z}^2} + m(f_z^2 f_{\bar{z}^2} + f_{\bar{z}}^2 f_{z^2} + 4f_z f_{\bar{z}} f_{z\bar{z}}) + m^2(f_z f_{\bar{z}})^2.$$

Since  $e^{mf} \neq 0$  and  $m \neq 0$ , the equation  $\Delta^2(e^{mf}) = 0$  is equivalent to  $A_m(z) = 0$ . In particular,  $A_2(z) - A_1(z) = 0$  and  $A_3(z) - A_2(z) = 0$ . That is,

$$\begin{cases} f_z^2 f_{\bar{z}^2} + f_{\bar{z}}^2 f_{z^2} + 4f_z f_{\bar{z}} f_{z\bar{z}} + 3(f_z f_{\bar{z}})^2 = 0, \\ f_z^2 f_{\bar{z}^2} + f_{\bar{z}}^2 f_{z^2} + 4f_z f_{\bar{z}} f_{z\bar{z}} + 5(f_z f_{\bar{z}})^2 = 0. \end{cases}$$

Subtracting the first equation from the second gives  $(f_z f_{\bar{z}})^2 = 0$ . Therefore,  $f$  is either analytic or anti-analytic. For any analytic function  $F$ ,  $F \circ f$  is then analytic or anti-analytic. Obviously,  $F \circ f \in \mathbb{H}_2$  which implies the desired statement of Proposition 2.  $\square$

## 3. THE PROOFS OF MAIN THEOREMS

3.1. **The proof of Theorem 1.** It suffices to prove the necessary parts of the statements (a) to (c), since the sufficiency parts are obvious. Let

$$f(z) = \sum_{k=1}^p |z|^{2(k-1)} G_k(z) \neq 0,$$

where each  $G_k(z)$  is harmonic.

(a) Assume that  $f \circ F \in \mathbb{H}_l$  for any analytic function  $F$ . Let  $F(z) = z^m$  ( $m \in \mathbb{Z}$ ,  $m > l$ ) be given. Then

$$f \circ F(z) = \sum_{k=1}^p |z|^{2m(k-1)} G_k(z^m) \in \mathbb{H}_l.$$

Obviously, each  $G_k(z^m)$  is still harmonic. Set  $t = \max\{k : G_k(z) \neq 0, 1 \leq k \leq p\}$ . We find that

$$\sum_{k=1}^p |z|^{2m(k-1)} G_k(z^m) = \sum_{k=1}^t |z|^{2m(k-1)} G_k(z^m) \in \mathbb{H}_{m(t-1)+1}^*.$$

Thus, by (1), we have that  $m(t-1)+1 \leq l$  which implies that  $t = 1$  and thus,  $f(z) = G_1(z)$  which is harmonic.

(b) Assume that  $f \circ F \in \mathbb{H}_l$  for any  $F \in \mathbb{H}_1^*$ . Suppose that  $F(z) = \bar{z}^m$  ( $m \in \mathbb{Z}$ ,  $m > l$ ) is given. Then, since for each  $k \in \{1, \dots, p\}$ ,  $G_k(\bar{z}^m) = \overline{G_k(z^m)}$  is harmonic, using the similar analysis of the previous case,  $f(z)$  must be harmonic. So, we may let

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \overline{\sum_{n=0}^{\infty} \beta_n z^n}.$$

Again, given  $F(z) = z^m + (c\bar{z})^m$  ( $m \in \mathbb{Z}$ ,  $m > l$ ,  $|c| = 1$ ), by a straight calculation, we compute that

$$\begin{aligned} \sum_{n=0}^{\infty} \alpha_n (F(z))^n &= \sum_{n=0}^{\infty} \alpha_n \left( \sum_{k=0}^n C_n^k z^{mk} (c\bar{z})^{m(n-k)} \right) \\ &= \sum_{i,j \geq 0} C_{i+j}^j \alpha_{i+j} z^{mi} (c\bar{z})^{mj} \\ &= \sum_{n=0}^{\infty} \alpha_n z^{mn} + \sum_{n=0}^{\infty} c^{mn} \alpha_n \bar{z}^{mn} + \sum_{i,j \geq 1} C_{i+j}^j \alpha_{i+j} z^{mi} (c\bar{z})^{mj} \\ &= \sum_{n=0}^{\infty} \alpha_n z^{mn} + \sum_{n=0}^{\infty} c^{mn} \alpha_n \bar{z}^{mn} + \sum_{j=1}^{\infty} C_{2j}^j c^{mj} \alpha_{2j} |z|^{2jm} \\ &\quad + \sum_{i > j \geq 1} C_{i+j}^j \alpha_{i+j} (c^{mj} z^{m(i-j)} + c^{mi} \bar{z}^{m(i-j)}) |z|^{2jm} \\ (2) \quad &= \sum_{n=0}^{\infty} \alpha_n z^{mn} + \sum_{n=0}^{\infty} c^{mn} \alpha_n \bar{z}^{mn} + \sum_{j=1}^{\infty} B_j(z) |z|^{2jm}, \end{aligned}$$

where

$$B_j(z) = C_{2j}^j c^{mj} \alpha_{2j} + \sum_{i>j}^{\infty} C_{i+j}^j \alpha_{i+j} (c^{mj} z^{m(i-j)} + c^{mi} \bar{z}^{m(i-j)})$$

for  $j \geq 1$ . Similarly, we have

$$(3) \quad \overline{\sum_{n=0}^{\infty} \beta_n (F(z))^n} = \sum_{n=0}^{\infty} \bar{\beta}_n \bar{z}^{mn} + \sum_{n=0}^{\infty} \bar{c}^{mn} \bar{\beta}_n z^{mn} + \sum_{j=1}^{\infty} C_j(z) |z|^{2jm},$$

where

$$C_j(z) = C_{2j}^j \bar{c}^{mj} \bar{\beta}_{2j} + \sum_{i>j}^{\infty} C_{i+j}^j \bar{\beta}_{i+j} (\bar{c}^{mj} \bar{z}^{m(i-j)} + \bar{c}^{mi} z^{m(i-j)})$$

for  $j \geq 1$ . Adding the last two expressions, namely, (2) and (3), gives

$$f \circ F(z) = \sum_{n=0}^{\infty} (\alpha_n + \bar{c}^{mn} \bar{\beta}_n) z^{mn} + \sum_{n=0}^{\infty} (c^{mn} \alpha_n + \bar{\beta}_n) \bar{z}^{mn} + \sum_{j=1}^{\infty} D_j(z) |z|^{2jm},$$

where  $D_j(z) = B_j(z) + C_j(z)$  for  $j \geq 1$ , and

$$\begin{aligned} D_j(z) &= C_{2j}^j (c^{mj} \alpha_{2j} + \bar{c}^{mj} \bar{\beta}_{2j}) \\ &\quad + \sum_{i>j}^{\infty} C_{i+j}^j \left( (c^{mj} \alpha_{i+j} + \bar{c}^{mi} \bar{\beta}_{i+j}) z^{m(i-j)} + (c^{mi} \alpha_{i+j} + \bar{c}^{mj} \bar{\beta}_{i+j}) \bar{z}^{m(i-j)} \right). \end{aligned}$$

Clearly, each  $D_j(z)$  is harmonic. Since  $f \circ F(z) \in \mathbb{H}_l$  and  $m > l$ , by (1), we have  $D_j(z) \equiv 0$  for each  $j \geq 1$ . It can be deduced from Parseval's formula that

$$c^{mj} \alpha_{2j} + \bar{c}^{mj} \bar{\beta}_{2j} = c^{mj} \alpha_{i+j} + \bar{c}^{mi} \bar{\beta}_{i+j} = c^{mi} \alpha_{i+j} + \bar{c}^{mj} \bar{\beta}_{i+j} \equiv 0$$

for each  $i > j \geq 1$  and every  $c$  with  $|c| = 1$ . It follows that

$$\alpha_{2j} = \beta_{2j} = \alpha_{i+j} = \beta_{i+j} = 0$$

for all  $i > j \geq 1$ . Thus,  $\alpha_n = \beta_n = 0$  for  $n \geq 2$  which yields that

$$f(z) = \alpha_0 + \alpha_1 z + \overline{\beta_0 + \beta_1 z}.$$

(c) Assume that  $f \circ F \in \mathbb{H}_l$  for any  $F \in \mathbb{H}_q^*$  ( $q \geq 2$ ).

We first claim that  $f$  is a harmonic polynomial and then we show that  $f$  is either a constant function or an affine mapping. For this, we begin to consider the representation (1) with

$$G_k(z) = h_k(z) + \overline{g_k(z)} \quad (1 \leq k \leq p),$$

where  $h_k(z) = \sum_{n=0}^{\infty} \alpha_{k,n} z^n$  and  $g_k(z) = \sum_{n=0}^{\infty} \beta_{k,n} z^n$ . Choosing

$$F(z) = c|z|^{2(q-1)} \quad (|c| = 1),$$

we find from (1) that

$$\begin{aligned}
f \circ F(z) &= \sum_{k=1}^p |z|^{4(k-1)(q-1)} \left( \sum_{n=0}^{\infty} c^n \alpha_{k,n} |z|^{2n(q-1)} + \overline{\sum_{n=0}^{\infty} c^n \beta_{k,n} |z|^{2n(q-1)}} \right) \\
&= \sum_{k=1}^p \left( \sum_{n=0}^{\infty} (c^n \alpha_{k,n} + \bar{c}^n \bar{\beta}_{k,n}) |z|^{2(n+2(k-1))(q-1)} \right) \\
&= \sum_{k=1}^p \left( \sum_{n=2(k-1)}^{\infty} (c^{n-2(k-1)} \alpha_{k,n-2(k-1)} + \bar{c}^{n-2(k-1)} \bar{\beta}_{k,n-2(k-1)}) |z|^{2n(q-1)} \right) \\
&= \sum_{n=0}^{\infty} A_n |z|^{2n(q-1)},
\end{aligned}$$

where

$$A_n = \begin{cases} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (c^{n-2k} \alpha_{k+1,n-2k} + \bar{c}^{n-2k} \bar{\beta}_{k+1,n-2k}) & \text{if } n < 2(p-1), \\ \sum_{k=0}^{p-1} (c^{n-2k} \alpha_{k+1,n-2k} + \bar{c}^{n-2k} \bar{\beta}_{k+1,n-2k}) & \text{if } n \geq 2(p-1). \end{cases}$$

Since  $f \circ F \in \mathbb{H}_l$ , by (1), we have  $A_n = 0$  for  $n > \lfloor \frac{l-1}{q-1} \rfloor$ .

Next we deduce that all analytic functions  $h_k(z)$  and  $g_k(z)$  should be polynomials. To do this, we fix  $n \in \mathbb{N}$  such that

$$n \geq \max \left\{ \left\lfloor \frac{l-1}{q-1} \right\rfloor, 2(p-1) \right\} + 1.$$

Let  $c = w^i$  ( $i = 0, 1, \dots, 2p-1$ ) in  $A_n = 0$ , where  $w$  is a primitive  $4pn$ -th root of unity. Then we have the following equations

$$\sum_{k=0}^{p-1} \left( (w^i)^{n-2k} \alpha_{k+1,n-2k} + (\bar{w}^i)^{n-2k} \bar{\beta}_{k+1,n-2k} \right) = 0, \quad i = 0, 1, \dots, 2p-1.$$

It is easy to see that the coefficient determinant of the above equations is a Vandermonde determinant. As  $w$  is a primitive  $4pn$ -th root of unity, it is clear that  $w^i \neq w^j$  ( $i \neq j$ ) and  $w^i \neq \bar{w}^j$  for  $i, j \in \{0, 1, \dots, n(2p-1)\}$ . Thus, this Vandermonde determinant does not vanish. It follows that

$$\alpha_{k+1,n-2k} = \bar{\beta}_{k+1,n-2k} = 0 \text{ for } k = 0, 1, \dots, p-1.$$

Therefore, all  $h_k(z)$  and  $g_k(z)$  ( $1 \leq k \leq p$ ) are polynomials. If  $p = 1$ , then it is clear that  $f$  is a harmonic polynomial. For  $p \geq 2$ , we will prove that  $f$  is also a harmonic polynomial.

Without loss of generality, we may let  $h_k(z) = \sum_{n=0}^d \alpha_{k,n} z^n$  and  $g_k(z) = \sum_{n=0}^d \beta_{k,n} z^n$  ( $d \geq 2, 1 \leq k \leq p$ ). Again, we choose

$$F(z) = z^{(m+1)(q-1)} \bar{z}^{q-1} = |z|^{2(q-1)} z^{m(q-1)} \in \mathbb{H}_q^* \quad (m \in \mathbb{Z}, m > l+d).$$

By computation, we have

$$\begin{aligned}
& f \circ F(z) \\
&= \sum_{k=1}^p \left| |z|^{2(q-1)} z^{m(q-1)} \right|^{2(k-1)} \left( \sum_{n=0}^d \alpha_{k,n} (|z|^{2(q-1)} z^{m(q-1)})^n + \overline{\sum_{n=0}^d \beta_{k,n} (|z|^{2(q-1)} z^{m(q-1)})^n} \right) \\
&= \sum_{k=1}^p \sum_{n=0}^d \left( \alpha_{k,n} z^{nm(q-1)} + \overline{\beta_{k,n}} \overline{z}^{nm(q-1)} \right) |z|^{2(t_k+n)(q-1)} \\
&= \sum_{k=1}^p \sum_{n=t_k}^{t_k+d} \left( \alpha_{k,n-t_k} z^{(n-t_k)m(q-1)} + \overline{\beta_{k,n-t_k}} \overline{z}^{(n-t_k)m(q-1)} \right) |z|^{2n(q-1)} \\
&= \sum_{n=0}^{(m+2)(p-1)+d} E_n(z) |z|^{2n(q-1)},
\end{aligned}$$

where  $t_k = (m+2)(k-1)$  for  $k \in \{1, \dots, p\}$  and

$$E_n(z) = \begin{cases} \alpha_{k,t} z^{tm(q-1)} + \overline{\beta_{k,t}} \overline{z}^{tm(q-1)} & \text{for } n = (m+2)(k-1) + t \ (t \in \{0, 1, \dots, d\}), \\ 0 & \text{otherwise,} \end{cases}$$

where  $k \in \{1, \dots, p\}$ . Obviously, all  $E_n(z)$ 's are harmonic polynomials. Since  $f \circ F(z) \in \mathbb{H}_l$  and  $(m+2)(k-1) > l$  for  $k \in \{2, \dots, p\}$ , by (1), we have  $E_n(z) \equiv 0$  for  $n \geq m+2$ . Again, by Parseval's formula, it follows that

$$\alpha_{k,t} = \beta_{k,t} = 0$$

for  $k \in \{2, \dots, p\}$  and  $t \in \{0, \dots, d\}$ . Therefore,  $f(z)$  is a harmonic polynomial and thus, for simplicity, we may write it as

$$f(z) = \sum_{n=0}^d (\alpha_n z^n + \overline{\beta_n} \overline{z}^n) \ (d > 1).$$

Now, we show that  $f(z)$  is either a constant function or an affine mapping. To do this, we choose  $F(z) = c|z|^{2(q-1)}$ , where  $|c| = 1$  and see that

$$f \circ F(z) = \sum_{n=0}^d (c^n \alpha_n + \overline{c}^n \overline{\beta_n}) |z|^{2n(q-1)} \in \mathbb{H}_l.$$

By the representation (1), we conclude that

$$c^n \alpha_n + \overline{c}^n \overline{\beta_n} = 0$$

for any  $n > \frac{l-1}{q-1} \geq [\frac{l-1}{q-1}] = t$  and  $c$  with  $|c| = 1$ . This gives  $\alpha_n = \beta_n = 0$  for  $n > t$ . Next we divide the proof into three cases.

(i) If  $t = 0$ , then  $l < q$  and thus,  $f(z) = \alpha_0 + \overline{\beta_0}$  which is a constant. Obviously,  $f \circ F(z)$  reduces to a constant and hence, belongs to  $\mathbb{H}_l$  for any  $F(z) \in \mathbb{H}_q^*$ .

(ii) If  $t = 1$ , then  $q \leq l \leq 2q-2$  and  $f(z) = \alpha_0 + \alpha_1 z + \overline{\beta_0 + \beta_1 z}$ . Obviously,  $f \circ F(z) \in \mathbb{H}_q \subset \mathbb{H}_l$  for any  $F(z) \in \mathbb{H}_q^*$ .



(iii) If  $t \geq 2$ , then  $l > 2q - 2$ . We claim that  $\alpha_n = \beta_n = 0$  for  $2 \leq n \leq t$ . Again, let

$$F(z) = c|z|^{2(q-1)}(z^{2m} + \bar{z}^{2m}) \quad (m \in \mathbb{Z}, \quad m > l, \quad |c| = 1).$$

Then, again by (1), we have

$$\begin{aligned} f \circ F(z) &= \sum_{n=0}^1 (c^n \alpha_n + \bar{c}^n \bar{\beta}_n) |z|^{2n(q-1)} (z^{2m} + \bar{z}^{2m})^n \\ &\quad + \sum_{n=2}^t (c^n \alpha_n + \bar{c}^n \bar{\beta}_n) |z|^{2n(q-1)} (z^{2m} + \bar{z}^{2m})^n \\ &= H(z) + \sum_{n=2}^t (c^n \alpha_n + \bar{c}^n \bar{\beta}_n) H_n(z), \end{aligned}$$

where

$$H(z) = \sum_{n=0}^1 (c^n \alpha_n + \bar{c}^n \bar{\beta}_n) |z|^{2n(q-1)} (z^{2m} + \bar{z}^{2m})^n \quad \text{and} \quad H_n(z) = |z|^{2n(q-1)} (z^{2m} + \bar{z}^{2m})^n.$$

Clearly,  $H(z) \in \mathbb{H}_q$ . However, if  $n \geq 2$  is an even number, then one writes

$$\begin{aligned} H_n(z) &= |z|^{2n(q-1)} \sum_{k=0}^n C_n^k z^{2m(n-k)} \bar{z}^{2mk} \\ &= |z|^{2n(q-1)} \left( C_n^{\frac{n}{2}} |z|^{2nm} + \sum_{k=0}^{\frac{n}{2}-1} (C_n^k z^{2m(n-2k)} |z|^{4mk} + C_n^{\frac{n}{2}+k+1} \bar{z}^{4m(k+1)} |z|^{2m(n-2-2k)}) \right), \end{aligned}$$

and thus, it follows from (1) that  $H_n(z) \in \mathbb{H}_{n(m+q-1)+1}^*$  and  $H_n(z)$  can be expressed as

$$H_n(z) = C_n^{\frac{n}{2}} |z|^{2n(m+q-1)} + \widetilde{H}_n(z),$$

where  $\widetilde{H}_n(z) \in \mathbb{H}_{n(m+q-1)}$ . If  $n \geq 2$  is an odd number, then it is easy to deduce that  $H_n(z) \in \mathbb{H}_{n(m+q-1)-m+1}^*$  and

$$H_n(z) = C_n^{\frac{n-1}{2}} (z^{2m} + \bar{z}^{2m}) |z|^{2(n(m+q-1)-m)} + \widehat{H}_n(z),$$

where  $\widehat{H}_n(z) \in \mathbb{H}_{n(m+q-1)-m}$ . Note that both  $n(m+q-1)+1$  and  $n(m+q-1)-m+1$  are strictly monotonically increasing as  $n$  from 2 to  $t$ . Moreover, there are no integers  $n_1, n_2 \in [2, t]$  such that  $n_1(m+q-1)+1 = n_2(m+q-1)-m+1$ . Therefore, if  $t$  is an even number, then  $f \circ F$  has the following form

$$f \circ F(z) = C_t^{\frac{t}{2}} (c^t \alpha_t + \bar{c}^t \bar{\beta}_t) |z|^{2t(m+q-1)} + L_t(z),$$

where  $L_t(z) \in \mathbb{H}_{t(m+q+1)}$ . Since  $f \circ F \in \mathbb{H}_l$  and  $t(m+q-1)+1 > l+1$ , by (1), we get

$$c^t \alpha_t + \bar{c}^t \bar{\beta}_t = 0,$$

for any  $c$  with modulus one. Thus,  $\alpha_t = \beta_t = 0$ . If  $t > 2$ , then  $t-1$  is a odd number. Therefore,  $f \circ F$  can be written as

$$f \circ F = C_{t-1}^{\frac{t-2}{2}} (c^{t-1} \alpha_{t-1} + \bar{c}^{t-1} \bar{\beta}_{t-1}) (z^{2m} + \bar{z}^{2m}) |z|^{2((t-1)(m+q-1)-m)} + L_{t-1}(z),$$

where  $L_{t-1}(z) \in \mathbb{H}_{(t-1)(m+q-1)-m}$ . Since  $f \circ F \in \mathbb{H}_l$  and  $(t-1)(m+q+1) - m > l+1$ , by (1), we have that

$$c^{t-1}\alpha_{t-1} + \bar{c}^{t-1}\bar{\beta}_{t-1} = 0,$$

for any  $c$  with modulus one. Thus,  $\alpha_{t-1} = \beta_{t-1} = 0$ . If  $t = 3$ , the proof is finished. If  $t > 3$ , we can similarly obtain that

$$\alpha_{t-2} = \beta_{t-2} = \cdots = \alpha_2 = \beta_2 = 0,$$

since  $n(m+q-1)+1$  and  $n(m+q-1)-m+1$  are greater than  $l+1$  for any  $n \in \{2, \dots, t\}$ .

If  $t$  is an odd number, then, by a similar analysis in the even case, we get that

$$\alpha_t = \beta_t = \cdots = \alpha_2 = \beta_2 = 0.$$

In other words, if  $f \circ F \in \mathbb{H}_l$  for any  $F \in \mathbb{H}_q^*$  ( $q \geq 2$ ), then  $f(z) = \sum_{n=0}^t \alpha_n z^n + \overline{\sum_{n=0}^t \beta_n z^n}$ , where  $t \leq \min\{1, [\frac{l-1}{q-1}]\}$ . The proof of Theorem 1 is finished.  $\square$

**3.2. The proof of Theorem 2.** The statement (a) is equivalent to proving

$$\mathbb{H}_{1,0}^l \cap \mathbb{H}_{1,1}^l = \{f : f \text{ is either analytic or anti-analytic}\}.$$

By Proposition 1, we have  $\mathbb{H}_{1,1}^l = \mathbb{H}_{1,0}^l$  and thus, it suffices to prove

$$\mathbb{H}_{1,0}^l = \{f : f \text{ is either analytic or anti-analytic}\}.$$

Let  $f \in \mathbb{H}_{1,0}^l$ . Then,  $f(z)$  is harmonic and thus, has the form  $f(z) = h(z) + \overline{g(z)}$ , where  $h(z)$  and  $g(z)$  are analytic. Set

$$F(z) = (e^z)^m \quad (m \in \mathbb{Z}, m > l).$$

Then we get that

$$F \circ f(z) = (e^{h(z)} \overline{e^{g(z)}})^m = H^m(z) \overline{G^m(z)},$$

where  $H(z) = e^{h(z)}$  and  $G(z) = e^{g(z)}$  are analytic. Since  $F \circ f \in \mathbb{H}_l$ , we see that

$$\Delta^l(F \circ f(z)) = \Delta^l(H^m(z) \overline{G^m(z)}) = 4^l \frac{\partial^l H^m(z)}{\partial z^l} \frac{\partial^l \overline{G^m(z)}}{\partial \bar{z}^l} = 0,$$

which yields

$$\text{either } \frac{\partial^l H^m(z)}{\partial z^l} = 0 \quad \text{or} \quad \frac{\partial^l G^m(z)}{\partial \bar{z}^l} = 0.$$

Thus, either  $H^m(z)$  or  $G^m(z)$  is a polynomial with degree not more than  $l-1$ . Note that  $m > l$ . Consequently, either  $H(z)$  or  $G(z)$  is a constant which in turn implies that either  $h(z)$  or  $g(z)$  is a constant. Therefore,  $f(z)$  is analytic or anti-analytic. Obviously, for any harmonic  $F$  and any analytic or anti-analytic  $f$ ,  $F \circ f$  is harmonic which is also  $l$ -harmonic.

(b) The statement (b) is equivalent to showing that

$$\mathbb{H}_{1,q}^l = \left\{ f : f = \sum_{n=0}^t \alpha_n z^n \quad \text{or} \quad f(z) = \overline{\sum_{n=0}^t \beta_n z^n} \right\},$$

where  $q \geq 2$  and  $t \leq [\frac{l-1}{q-1}]$ . By Proposition 1, we have  $\mathbb{H}_{1,q}^l \subseteq \mathbb{H}_{1,0}^l$ . Thus, if  $f \in \mathbb{H}_{1,q}^l$ , then either  $f$  is analytic or anti-analytic.

Since  $F \circ \bar{f} = \overline{F \circ f}$ , we only need to consider the case that  $f$  is analytic. Let  $F(z) = |z|^{2(q-1)} (\in \mathbb{H}_q^*)$ . Then we have that

$$H(z) = F \circ f(z) = f^{q-1}(z) \bar{f}^{q-1}(z).$$

If  $H \in \mathbb{H}_l$ , then we have

$$\Delta^l H(z) = 4^l \frac{\partial^l f^{q-1}(z)}{\partial z^l} \frac{\partial^l \bar{f}^{q-1}(z)}{\partial \bar{z}^l} = 0,$$

which yields  $\frac{\partial^l f^{q-1}(z)}{\partial z^l} = 0$ . Therefore,  $f^{q-1}(z) = \sum_{n=0}^{l-1} \beta_n z^n$  which implies that  $f$  must be a polynomial of degree not more than  $\lfloor \frac{l-1}{q-1} \rfloor$  and thus, we write  $f(z) = \sum_{n=0}^t \alpha_n z^n$  ( $t \leq \lfloor \frac{l-1}{q-1} \rfloor$ ). For any  $F \in \mathbb{H}_q^*$  ( $q \geq 2$ ),  $F$  has the representation

$$F(z) = \sum_{k=1}^q |z|^{2(k-1)} G_k(z),$$

where each  $G_k(z)$  is harmonic and  $G_q(z) \neq 0$ . By computation, we may then write

$$\begin{aligned} F \circ f(z) &= \sum_{k=1}^q \left( \sum_{n=0}^t \alpha_n z^n \right)^{k-1} \overline{\left( \sum_{n=0}^t \alpha_n z^n \right)^{k-1}} G_k(f(z)) \\ &= \sum_{k=1}^q \left( \sum_{n=0}^{t(k-1)} c_n z^n \right) \left( \sum_{n=0}^{t(k-1)} \bar{c}_n \bar{z}^n \right) G_k(f(z)) \quad \text{for some } c_n \text{'s} \\ &= \sum_{k=1}^q \left( \sum_{0 \leq i, j \leq t(k-1)} C_{ij} z^i \bar{z}^j G_k(f(z)) \right), \end{aligned}$$

where each  $C_{i,j}$  is a complex number. It is easy to deduce that  $z^i \bar{z}^j G_k(f(z))$  is  $\max\{i+1, j+1\}$ -harmonic. Since  $i, j \leq t(q-1)$ , we see that  $F \circ f \in \mathbb{H}_{t(q-1)+1}$ . As  $t(q-1)+1 \leq l$ , it is obvious that  $F \circ f \in \mathbb{H}_l$ . The proof is complete.  $\square$

**3.3. The proof of Theorem 3.** The sufficiency parts of the statements in (a)-(c) are obvious and therefore, we need to prove only the necessary part of the theorem.

(a) The statement (a) is equivalent to proving

$$\mathbb{H}_{p,0}^1 \cap \mathbb{H}_{p,1}^1 = \mathbb{H}_{p,0}^2 \cap \mathbb{H}_{p,1}^2 = \{f : f \text{ is either analytic or anti-analytic}\},$$

where  $p \geq 2$ . By Proposition 1 and Theorem 2, we acquire

$$\mathbb{H}_{p,1}^1 = \mathbb{H}_{p,0}^1 = \mathbb{H}_{1,0}^1 = \{f : f \text{ is either analytic or anti-analytic}\}.$$

By Propositions 1 and 2, we get

$$\mathbb{H}_{p,1}^2 = \mathbb{H}_{p,0}^2 = \mathbb{H}_{2,0}^2 = \{f : f \text{ is either analytic or anti-analytic}\}.$$

(b) By Proposition 1, we have  $\mathbb{H}_{p,q}^1 = \mathbb{H}_{1,q}^1$ . Therefore, the statement is an immediate consequence of Theorem 2.

(c) By Proposition 1, we obtain that  $\mathbb{H}_{p,q}^2 = \mathbb{H}_{2,q}^2 \subseteq \mathbb{H}_{2,0}^2$ . By Proposition 2, it follows that  $\mathbb{H}_{p,q}^2 \subseteq \mathbb{H}_{1,q}^2$  and the desired conclusion follows.  $\square$

**Acknowledgments.** The research was supported by the NSF's of China (No.11371363), the construct program of the key discipline in Hunan province and the Hunan Provincial Natural Science Foundation of China (No.2015JJ6011).

## REFERENCES

- [1] Z. Abdulhadi and Y. Abu Muhanna, *Landau's theorem for biharmonic mappings*, J. Math. Anal. Appl., **338** (2008), 705–709.
- [2] Z. Abdulhadi, Y. Abu Muhanna and S. Khuri, *On univalent solutions of the biharmonic equation*, J. Inequal. Appl., **5** (2005), 469–478.
- [3] Z. Abdulhadi, Y. Abu Muhanna and S. Khoury, *On some properties of solutions of the biharmonic equation*, Appl. Math. Comput., **177** (2006), 346–351.
- [4] N. Aronszajn, T.M. Creese and L.J. Lipkin, *Polyharmonic functions*, Clarendon Press, Oxford, 1983.
- [5] A. Borichev and H. Hedenmalm, *Weighted integrability of polyharmonic functions*, Adv. Math., **264**(2014), 464–505.
- [6] SH. Chen, S. Ponnusamy and X. Wang, *Landau's theorem for certain biharmonic mappings*, Appl. Math. Comput., **208** (2009), 427–433.
- [7] SH. Chen, S. Ponnusamy and X. Wang, *Compositions of harmonic mappings and biharmonic mappings*, Bull. Belg. Math. Soc. Simon Stevin, **17**(2010), 693–704.
- [8] SH. Chen, S. Ponnusamy and X. Wang, *Bloch and Landau's theorems for planar  $p$ -harmonic mappings*, J. Math. Anal. Appl., **373** (2011), 102–110.
- [9] SH. Chen, S. Ponnusamy and X. Wang, *On some properties of solutions of the  $p$ -harmonic equations*, Filomat, **27** (2013), 577–591.
- [10] J. Chen, A. Rasila and X. Wang, *On polyharmonic univalent mappings*, Math. Rep., **15** (2013), 343–357.
- [11] J. Chen, A. Rasila and X. Wang, *Starlikeness and convexity of polyharmonic mappings*, Bull. Belg. Math. Soc. Simon Stevin., **21** (2014), 67–82.
- [12] J. Chen, A. Rasila and X. Wang, *Landau's theorem for polyharmonic mappings*, J. Math. Anal. Appl., **409** (2014), 934–945.
- [13] J. Chen and X. Wang, *On certain classes of biharmonic mappings defined by convolution*, Abstr. Appl. Anal., 2012, Article ID 379130, 10 pages. doi:10.1155/2012/379130.
- [14] J. G. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. **9** (1984), 3–25.
- [15] P. Duren, *Harmonic mappings in the plane*, Cambridge University Press, Cambridge, 2004.
- [16] J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics*, Princeton-Hall, 1965.
- [17] S. A. Khuri, *Biorthogonal series solution of Stokes flow problems in sectorial regions*, SIAM J. Appl. Math., **56** (1996), 19–39.
- [18] S. A. Khuri, *On the properties of a class of polyharmonic functions*, Math. Model. Anal., **18** (2013), 219–235.
- [19] W. E. Langlois, *Slow Viscous Flow*, Macmillan Company, 1964.
- [20] P. Li, S. A. Khuri, X. Wang, *On certain geometric properties of polyharmonic mappings*, J. Math. Anal. Appl., **434** (2016), 1462–1473.
- [21] P. Li, S. Ponnusamy and X. Wang, *Some properties of planar  $p$ -harmonic and log- $p$ -harmonic mappings*, Bull. Malaysian Math. Sciences Soc., **36** (2013), 595–609.
- [22] J. Qiao and X. Wang, *Subordination of  $p$ -harmonic mappings*, Bull. Belg. Math. Soc. Simon Stevin, **19** (2012), 47–61.
- [23] J. Qiao and X. Wang, *On  $p$ -harmonic univalent mappings (in Chinese)*, Acta Math. Sci., **32** (2012), 588–600.
- [24] E. Reich, *The composition of harmonic mappings*, Ann. Acad. Sci. Fenn. Ser. A. I, **12** (1987), 47–53.

GANG LIU, COLLEGE OF MATHEMATICS AND STATISTICS (HUNAN PROVINCIAL KEY LABORATORY OF INTELLIGENT INFORMATION PROCESSING AND APPLICATION), HENGYANG NORMAL UNIVERSITY, HENGYANG, HUNAN 421008, PEOPLE'S REPUBLIC OF CHINA.

*E-mail address:* `liugangmath@sina.cn`

SAMINATHAN PONNUSAMY, INDIAN STATISTICAL INSTITUTE (ISI), CHENNAI CENTRE, SETS (SOCIETY FOR ELECTRONIC TRANSACTIONS AND SECURITY), MGR KNOWLEDGE CITY, CIT CAMPUS, TARAMANI, CHENNAI 600 113, INDIA.

*E-mail address:* `samy@isichennai.res.in`, `samy@iitm.ac.in`